TENTAMEN 22 april 2008
STATISTISCHE FYSICA

8.8   +0,8 $\boxed{9\frac{1}{2}}$

[redacted]

opgave 1+2 volledig
3+4a+4b+5
grotendeel
goed

1)

9

a. $$\Omega(n) = m^n \frac{N!}{n!(N-n)!}$$

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b. (bereken : $n = \ldots T$)

$\cancel{\frac{1}{T} = \frac{k}{\varepsilon} \ln\left(\frac{N-n}{n}\right)}$

$$E = n\varepsilon$$

$$S = k \ln \Omega$$

$$\frac{1}{T} = \frac{dS}{dE} = \frac{dS}{dn} \frac{dn}{dE}$$

$$\frac{dn}{dE} = \frac{d}{dE}\left(\frac{E}{\varepsilon}\right) = \frac{1}{\varepsilon} \quad (1)$$

$$\frac{dS}{dn} = \frac{d}{dn}(k \ln \Omega) = \frac{d}{dn}\left[k\left\{\ln\left(m^n \frac{N!}{n!(N-n)!}\right)\right\}\right]$$

$$= \frac{d}{dn}\left[k\left\{\ln m^n + \ln(N!) - \ln(n!) - \ln((N-n)!)\right\}\right]$$

$\ln N! = N \ln N - N$
$N \gg 1$

$$= \frac{d}{dn}\left[k\left\{n \ln m + N \ln N - \cancel{N} - n \ln n + n - (N-n)\ln(N-n) \cancel{+ (N-n)}\right\}\right]$$

$$= k\left\{\ln m - \cancel{n} \cdot \frac{1}{n} - \ln n + \frac{\cancel{(N-n)}}{\cancel{(N-n)}} + \ln(N-n)\right\}$$

$$\frac{dS}{dn} = k \ln\left(\frac{N-n}{n} m\right) \quad (2)$$

$$(1)+(2): \quad \frac{1}{T} = \frac{k}{\varepsilon} \ln\left(\frac{N-n}{n} \cdot m\right)$$

vervolg opg 1b.

$$\frac{1}{T} = \frac{k}{\varepsilon} \ln\left(\frac{N-n}{n} m\right)$$

$$e^{\frac{\varepsilon}{kT}} = \frac{N-n}{n} \cdot m = \left(\frac{N}{n} - 1\right) m = \frac{Nm}{n} - m$$

$$\frac{1}{n} = \frac{e^{\frac{\varepsilon}{kT}} + m}{Nm}$$

$$n = \frac{Nm}{e^{\varepsilon/kT} + m}$$

c) $$C_v = \frac{dE}{dT} =$$

$$E = n\varepsilon$$

$$C_V = \frac{d(n\varepsilon)}{dT} = \varepsilon \frac{d}{dT}\left(\frac{Nm}{e^{\varepsilon/kT} + m}\right)$$

$$= Nm\varepsilon \frac{-1}{(e^{\varepsilon/kT} + m)^2} \cdot -\frac{\varepsilon}{kT^2} \cdot e^{\varepsilon/kT}$$

$$C_V = \frac{Nm\varepsilon^2 e^{\varepsilon/kT}}{kT^2 (e^{\varepsilon/kT} + m)^2}$$

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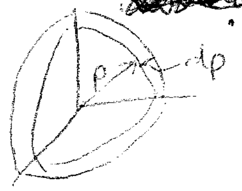
9

opg 2 / 2)

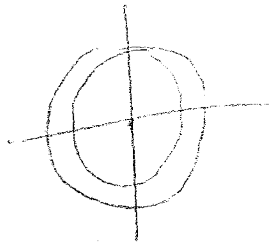


a. $$f(p)\,dp = V\,\frac{4\pi p^2\,dp}{h^3}$$

in 2-D : $\overbrace{4\pi p^2 dp}^{\cancel{opp}\ opp} \rightarrow \overbrace{2\pi p}^{omtrek} dp$

$V \rightarrow A$

$h^3 \rightarrow h^2$



$$\boxed{f_{2D}(p)\,dp = \frac{A\,2\pi p\,dp}{h^2}}$$

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$$Z_1 = \int_0^\infty f(p)\,dp\, e^{-\beta p^2/2m}$$

$$= \int_0^\infty e^{\frac{-\beta p^2}{2m}} \frac{A\,2\pi p}{h^2}\,dp$$

$$= \left[-\frac{A\,2\pi}{h^2} mkT\, e^{-\frac{\beta p^2}{2m}}\right]_0^\infty$$

$$= \frac{A\,2\pi mkT}{h^2}$$

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$$\left(\begin{aligned} & -\frac{2p\beta}{2m}\, e^{-\frac{\beta p^2}{2m}} \\ & = -\frac{p}{mkT}\, e^{-\frac{\beta p^2}{2m}} \\ & \frac{\partial}{\partial p}\left(-\frac{A 2\pi mkT}{h^2}\, e^{-\frac{\beta p^2}{2m}}\right) \\ & = \cancel{-}\,\frac{A\,2\pi mkT}{h^2}\, e^{-\frac{\beta p^2}{2m}} \\ & \cdot\ \cancel{-}\,\frac{2p}{2mkT} \end{aligned} \right)$$

$$Z = \frac{1}{N!}(Z_1)^N$$

$$Z = \frac{1}{N!}\left(\frac{A\,2\pi mkT}{h^2}\right)^N = \frac{1}{N!}A^N\left(\frac{2\pi mkT}{h^2}\right)^N$$

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Opg. 2

b $\quad F = E - TS$

$$\begin{cases} dF = dE - TdS - SdT \\ dE = dQ + dW = \overset{TdS}{\cancel{SdT}} - pdV \end{cases}$$

$$\left.\begin{aligned} dS &= \frac{dQ}{T} \\ dW &= -pdV \end{aligned}\right\} \text{reversib}$$

$$\to \quad dF = TdS - pdV - TdS - SdT$$

& $\quad dF = -SdT - pdV$

c. $\quad S = -\left(\frac{dF}{dT}\right)_V$

$$\left.\begin{aligned} F &= -kT \ln Z \\ Z &= \frac{1}{N!} A^N \left(\frac{2\pi m k T}{h^2}\right)^N \end{aligned}\right\} F = -kT\left\{\overset{-N\ln N + N}{\ln\left(\frac{1}{N!}\right)} + N\ln A + N\ln\left(\frac{2\pi m kT}{h^2}\right)\right.$$

$$= \ln\left(\frac{2\pi m k}{h^2}\right) + \ln T$$

$$F = -NkT\left\{1 - \ln N + \ln A + \ln\left(\frac{2\pi m k T}{h^2}\right)\right\}$$

$$\frac{dF}{dT} = -Nk\left\{1 - \ln N + \ln A + \ln\left(\frac{2\pi m kT}{h^2}\right)\right\} \cancel{+} \cancel{NkT}$$

$$-NkT \cdot \frac{1}{\cancel{T}}$$

$$S = -\left(\frac{dF}{dT}\right)_V = Nk\left\{1 + \ln A - \ln N + \ln\left(\frac{2\pi m k}{h^2}\right) + \ln T + 1\right\}$$

& $$= Nk\left\{\ln\frac{A}{N} + \ln T + 2 + \ln\left(\frac{2\pi m k}{h^2}\right)\right\}$$

Opg 3/8

3)

[redacted]

$p = 2 \text{ bar} = 2 \cdot 10^5 \text{ Pa}$

* ideaal gas: $pV = nkT = RT$

* water in evenwicht met waterdamp $\rightarrow T = T_k$

* $L = 2{,}25 \text{ MJ/kg}$

$$\frac{dp}{dT} = \frac{L}{T\Delta V}$$

* $p = 1 \text{ bar} \rightarrow T_k = 373 \text{ K}$

$$dp = \frac{L}{T\Delta V}\, dT = \frac{Lp}{RT^2}\, dT$$

$$\boxed{\Delta V = V_{gas} - V_{liquid} \simeq V_{gas} = \frac{RT}{p}}$$

$$\int_{p_1}^{p_2} \frac{1}{p}\, dp = \int_{T_1}^{T_2} \frac{L}{RT^2}\, dT$$

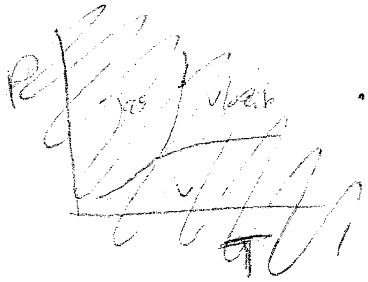
$p_1 = 1 \text{ bar}$ $\quad T_1 = 373 \text{ K}$
$p_2 = 2 \text{ bar}$ $\quad T_2 = ?$

$$\ln\left(\frac{p_2}{p_1}\right) = \left.\frac{-L}{RT}\right]_{T_1}^{T_2} = \frac{-L}{RT_2} + \frac{L}{RT_1}$$

$$\ln(2) = \cancel{\frac{2{,}25 \cdot 10^6}{8{,}31 \cdot 373}} \frac{L}{RT_1} - \frac{L}{RT_2} = \frac{L}{R}\left(\frac{1}{T_1} - \frac{1}{T_2}\right.$$

Vervolg Opg 3)

$$\ln(2) = \frac{L}{R}\left(\frac{1}{T_1} - \frac{1}{T_2}\right)$$



$$\frac{1}{T_1} - \frac{R}{L}\ln(2) = \frac{1}{T_2}$$

J/mol

$$\frac{1}{373} - \frac{8{,}31}{2{,}25\cdot10^{6}}\ln(2) = 2{,}68\cdot10^{-3}$$

J/kg → omrekenen naar J/mol

$$\rightarrow \quad T_2 = \frac{1}{2{,}68\cdot10^{-3}} = \boxed{373{,}4\ \mathrm{K}}$$

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Met omrekening naar J/mol
geeft $T_2 \approx 394\ \mathrm{K}$

Opg. 4

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4) a. photongas → Bosonen $n_r = 0, 1, 2, \ldots$

$E = n_1\varepsilon_1 + n_2\varepsilon_2 + \ldots$

$$Z_{ph} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{n_3=0}^{\infty} \cdots \; e^{-\beta(n_1\varepsilon_1 + n_2\varepsilon_2 + \ldots)} = \sum_{n_1=0}^{\infty} e^{-\beta n_1 \varepsilon_1} \sum_{n_2=0}^{\infty} e^{-\beta n_2 \varepsilon_2} \cdots$$

$$Z_{ph} = \prod_{r=1}^{\infty}\left(\sum_{n_r=0}^{\infty} e^{-\beta n_r \varepsilon_r}\right)$$

$$\sum_{i=0}^{\infty} x^i = \frac{1}{1-x}$$

$$\boxed{x < 1}$$

$$Z_{ph} = \prod_{r=1}^{\infty}\left(\sum_{n_r=0}^{\infty} \left(e^{-\beta\varepsilon_r}\right)^{n_r}\right)$$

$e^{-\beta\varepsilon_r} < 1$, want $\beta > 0$ en $\varepsilon_r \geq 0$ dus $e^{-\beta\varepsilon_r} < 1$.

$$Z_{ph} = \prod_{r=1}^{\infty}\left(\frac{1}{1 - e^{-\beta\varepsilon_r}}\right)$$

3

$\ln(ABC) = \ln A + \ln B + \ln C$

b) $F = -kT \ln Z$

$$= -kT \ln\left[\prod_{r=1}^{\infty} \frac{1}{1-e^{-\beta\varepsilon_r}}\right]$$

$$= -kT \sum_{r=1}^{\infty} \ln\left(\frac{1}{1-e^{-\beta\varepsilon_r}}\right)$$

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$$= kT \sum_{r=1}^{\infty} \ln\left(1 - e^{-\beta\varepsilon_r}\right)$$

$$= kT \int f(p)\, dp \, \ln\left(1 - e^{-\beta\varepsilon_r}\right)$$

Opg 4c)

$\left(\varepsilon_r = \hbar\omega\left(r+\frac{1}{2}\right)\right)$ $\quad pV = RT \rightarrow p = \frac{RT}{V}$

$T = \frac{pV}{R}$

c)

$p = T\left(\frac{\partial S}{\partial V}\right)_{N,E}$

$F = E - TS \quad dF = -SdT - pdV$

~~$dF = dE - TdS - SdT$~~ $\quad F = kT\sum_{r=1}^{\infty}\ln\left(1-e^{-\beta\varepsilon_r}\right)$

$F = -kT\ln Z$

$-kT\ln Z = E - TS \qquad \beta = \frac{1}{kT}$

$S = \frac{E}{T} + k\ln Z$

$= \frac{\sum_{r=1}^{\infty} n_r\varepsilon_r}{T} + k\sum_{r=1}^{\infty}\ln\left(1-e^{-\beta\varepsilon_r}\right)$

$\boxed{p = -\left(\frac{dF}{dV}\right)_T} \qquad F = kT\sum_{r=1}^{\infty}\ln\left(1-e^{-\beta\varepsilon_r}\right)$

toestandsdichtheid $\rightarrow$ zie b.

$= -\left(\frac{dF}{dT}\frac{dT}{dV}\right) = kT\frac{d\ln Z}{dT}\frac{dT}{dV}$

$\frac{dT}{dV} = \frac{p}{R}$, $\quad p = RT/V$

$= \frac{kTp}{R}\frac{\partial}{\partial T}\sum_{r=1}^{\infty}\ln\left(1-e^{-\beta\varepsilon_r}\right) \qquad \frac{d\varepsilon}{dx} = \frac{1}{\beta}$

$= \frac{kT^2}{V}\sum_{r=1}^{\infty}\frac{1}{1-e^{-\beta\varepsilon_r}}\cdot e^{-\beta\varepsilon_r}\cdot\frac{\varepsilon_r}{kT^2} \qquad x = \beta\varepsilon_r \quad \int\frac{x}{e^x-1}\frac{1}{\beta}dx$

$= \frac{kT^2}{V}\sum_{r=1}^{\infty}\frac{\varepsilon_r e^{-\beta\varepsilon_r}}{kT^2\left(1-e^{-\beta\varepsilon_r}\right)} = \frac{kT}{V}\underbrace{\sum_{r=1}^{\infty}\frac{\beta\varepsilon_r}{\beta\left(e^{\beta\varepsilon_r}-1\right)}}_{=\frac{\pi^2}{6}\cdot\frac{1}{\beta}}$

$p = \frac{kT\pi^2}{6V}\cdot\frac{1}{\beta} = \frac{k^2T^2\pi^2}{6V}$

Opg 5

5)

a). $n(\varepsilon) = \dfrac{1}{e^{\beta(\varepsilon-\mu)} - 1}$ $e^x - 1 \approx x \quad x \ll 1$

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$$f(p)\,dp = \frac{V\,4\pi p^2\,dp}{h^3}$$

$\varepsilon = \dfrac{p^2}{2m}$ $\dfrac{dp}{d\varepsilon} = \dfrac{m}{\sqrt{2m\varepsilon}}$ $p = \sqrt{2m\varepsilon}$

$$f(\varepsilon)\,d\varepsilon = \frac{V\,4\pi \cdot 2m\varepsilon}{h^3}\,\frac{m}{\sqrt{2m\varepsilon}}\,d\varepsilon$$

(3 pnt voor a)

$\beta(\varepsilon-\mu)$

$(\beta\varepsilon - \beta\mu)^{3/2}$

$$f(\varepsilon)\,d\varepsilon = \frac{V\,2\pi}{h^3}\,(2m)^{3/2}\,\varepsilon^{1/2}\,d\varepsilon$$

$$N = \int_0^\infty n(\varepsilon)\,f(\varepsilon)\,d\varepsilon = \int_0^\infty \frac{V\,2\pi}{h^3}\,(2m)^{3/2}\,\frac{\varepsilon^{1/2}}{e^{\beta(\varepsilon-\mu)} - 1}\,d\varepsilon$$

$$\approx \frac{V\,2\pi}{h^3}\,(2m)^{3/2}\int_0^\infty \frac{(\beta\varepsilon)^{1/2}}{\beta^{3/2}\left(e^{\beta\varepsilon} - 1\right)}\,d\varepsilon$$

$e^{\beta(\varepsilon-\mu)} - 1 \approx (e^{\beta\varepsilon} - 1)\beta$

$\mu < 0$
$\mu \ll \varepsilon$

$$N = \frac{V\,2\pi}{h^3}\,(2m)^{3/2}\,\frac{1}{\beta^{3/2}} \cdot 2{,}612\,\frac{\sqrt{\pi}}{2}$$

$$N = \frac{V\,(2m\pi)^{3/2}\,(kT)^{3/2}}{h^3} \cdot 2{,}612$$

vervolg opg 5a.

$$N = \frac{V\,(2\pi m)^{3/2}\,(kT)^{3/2}\;2{,}612}{h^3}$$

$$N^2 = \frac{V^2\,(2\pi m)^3\,(kT)^3\,(2{,}612)^2}{h^6}$$

$$T^3 = \frac{N^2 h^6}{V^2\,(2\pi m)^3\,(2{,}612)^2\,k^3}$$

toelichting?!

$$T_c = \frac{h^2}{2\pi m k}\left(\frac{N}{2{,}612\,V}\right)^{2/3}$$

$$N = n_1 + \int_0^\infty n(\varepsilon) f(\varepsilon)\, d\varepsilon$$

$$n_1 = N - \int_0^\infty \ldots > 0$$

geeft

$$\frac{h^2}{2\pi m k}\left(\frac{N}{2{,}612\,V}\right)^{2/3} > T$$

dus

$$T_c = \frac{h^2}{2\pi m k}\left(\frac{N}{2{,}612\,V}\right)^{2/3}$$

b) ρ (He liquid) $= 124\ \mathrm{g/dm^3}$

$$\rho = \frac{m}{V} = 124\ \mathrm{g/dm^3} = 124\ \mathrm{kg/m^3}$$

$(\cdot 10^{-3}\ \ 1\cdot 10^{3})$

$h = 6{,}626\cdot 10^{-34}$ Js

$m^{(\text{per mol})} = (2\cdot 9{,}11\cdot 10^{-31} + 2\cdot 1{,}62\cdot 10^{-27})\,\mathrm{kg}\cdot 6{,}02\cdot 10^{23}$

$= 1{,}952\cdot 10^{-3}$

$k = 1{,}381\cdot 10^{23}\ \mathrm{JK^{-1}}$

$$T_c = \frac{(6{,}626\cdot 10^{-34})^2}{2\cdot\pi\cdot(2\cdot 9{,}11\cdot 10^{-31} + 2\cdot 1{,}62\cdot 10^{-27})\cdot 1{,}381\cdot 10^{23}}\left(\frac{\frac{6{,}02\cdot 10^{23}}{2{,}612}\cdot 124}{1{,}952\cdot 10^{-3}}\right)^{2/3}$$

2

$= 9{,}3\cdot 10^{-46}$ K

niet echt plausibel ......

waarde van k verkeerd gegeven op tentamen: 10^{23} ipv 10^{-23}

c) $\rho\,(\text{liquid}\,H_2) = 70{,}8\ g/dm^3 = 70{,}8\ \frac{kg}{m^3}$

~~$m_{H_2} = (2 \cdot 9{,}11 \cdot 10^{-31} + 2 \cdot 1{,}67 \cdot 10^{-27})\ kg$~~

~~$= 70{,}8\ \frac{kg}{m^3}$~~

$k = 1{,}381 \cdot 10^{23}\ J\,K^{-1}$

$h = 6{,}626 \cdot 10^{-34}\ J \cdot s$

$N = 6{,}02 \cdot 10^{23}\ mol^{-1}$

$= 2{,}01 \cdot 10^{-3}\ k$

3 $m\,(1\ mol\ H_2) = (2 \cdot 9{,}11 \cdot 10^{-31} + 2 \cdot 1{,}67 \cdot 10^{-27}) \cdot 6{,}02 \cdot 10^{23}\ kg$

$$V = \frac{m}{\rho} = \frac{(2 \cdot 9{,}11 \cdot 10^{-31} + 2 \cdot 1{,}67 \cdot 10^{-27}) \cdot 6{,}02 \cdot 10^{23}}{70{,}8}\ m^3$$

$V_m = 2{,}84 \cdot 10^{-5}\ m^3$

$$T_c = \frac{h^2}{2\pi m k}\left(\frac{N}{2{,}612\,V}\right)^{2/3}$$ invullen geeft:

2 massa van 1 deeltje H_2

$T_c = 1{,}05 \cdot 10^{-44}\ K$

Met deze uitkomst is het niet mogelijk dat er Bose-Einstein Condensatie optreedt in vloeibaar waterstof want $T_c \ll 14K$, maar aangezien het antwoord bij opg b. ook zo'n kleine T_c gaf zal er ergens een reken/denk/typfout zitten in de opgave ~~berekeningen~~. ~~Zolang~~ Als $T_c < 14K$ zal er geen BE condensatie optreden in H_2 (liquid). Als $T_c > 14K$ dan bestaat er wel de mogelijkheid dat het gebeurt